A tropical analogue of the Hessian group

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Abstract

We investigate a tropical analogue of the Hessian group G_{216}, the group of linear automorphisms acting on the Hesse pencil. Through the procedure of ultradiscretization, the group law on the Hesse pencil reduces to that on the tropical Hesse pencil. We then show that the dihedral group $\mathcal{D}_3$ of degree three is the group of linear automorphisms acting on the tropical Hesse pencil.

The Hessian group $G_{216} \simeq \Gamma \rtimes SL(2,\mathbb{F}_3)$ is a subgroup of $PGL(3,\mathbb{C})$, the group of linear transformations on the projective plane $\mathbb{P}^2(\mathbb{C})$, where $\Gamma = (\mathbb{Z}/3\mathbb{Z})^2$ and $SL(2,\mathbb{F}_3)$ is the special linear group over the finite field $\mathbb{F}_3$ of characteristic three. The Hessian group is generated by the following four linear transformations

$$g_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{pmatrix} \qquad g_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{pmatrix} \qquad g_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix},$$

where ζ_3 denotes the primitive third root of 1. The name, "Hessian" group, comes from the fact that G_{216} is the group of linear automorphisms acting on the Hesse pencil [1, 4]. The Hesse pencil is a one-dimensional linear system of plane cubic curves in $\mathbb{P}^2(\mathbb{C})$ given by

$$f(x_0, x_1, x_2; t_0, t_1) := t_0 \left(x_0^3 + x_1^3 + x_2^3\right) + t_1 x_0 x_1 x_2 = 0,$$

where (x_0, x_1, x_2) is the homogeneous coordinate of $\mathbb{P}^2(\mathbb{C})$ and the parameter (t_0, t_1) ranges over $\mathbb{P}^1(\mathbb{C})$ [1]. The curve composing the pencil is called the Hesse cubic curve (see figure 1).

Each member of the pencil is denoted by E_{t_0,t_1} and the pencil itself by $\{E_{t_0,t_1}\}_{(t_0,t_1)\in\mathbb{P}^1(\mathbb{C})}$. The nine base points of the pencil are given as follows

$$\begin{array}{lll} p_0 = (0,1,-1) & p_1 = (0,1,-\zeta_3) & p_2 = (0,1,-\zeta_3^2) \\ p_3 = (1,0,-1) & p_4 = (1,0,-\zeta_3^2) & p_5 = (1,0,-\zeta_3) \\ p_6 = (1,-1,0) & p_7 = (1,-\zeta_3,0) & p_8 = (1,-\zeta_3^2,0). \end{array}$$

Any smooth curve in the pencil has the nine base points as its inflection points, and hence they are in the Hesse configuration [1, 4]. We choose p_0 as the unit of addition of the points on the Hesse cubic curve.

The group $E_{t_0,t_1}[3]$ of three torsion points on E_{t_0,t_1} consists of the nine base points $p_0, p_1, \cdots, p_8$. The map

$$p_1 \longmapsto (1,0) \qquad p_3 \longmapsto (0,1)$$

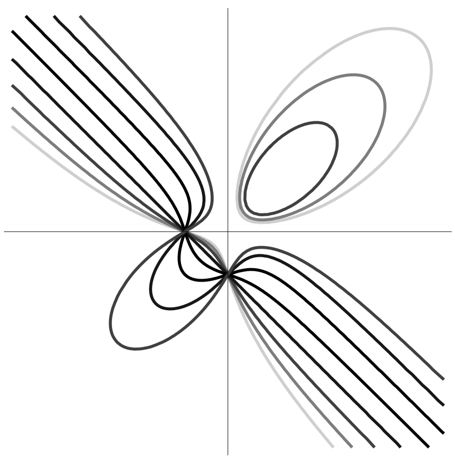

Figure 1: Several members of the Hesse pencil.

induces the group isomorphism $E_{t_0,t_1}[3] \simeq \Gamma$, which is the normal subgroup of G_{216} generated by the elements g_1 and g_2. Therefore, the action of Γ fixes the parameter (t_0, t_1) of the Hesse pencil. Let $\alpha : G_{216} \to PGL(2, \mathbb{C})$ be a map given by

$$\alpha(g): \ (t_0, t_1) = (x_0x_1x_2, x_0^3 + x_1^3 + x_2^3) \longmapsto (t_0', t_1') = (x_0'x_1'x_2', x_0'^3 + x_1'^3 + x_2'^3),$$

where $g \in G_{216}$ and $g : (x_0, x_1, x_2) \mapsto (x_0', x_1', x_2')$. Then we have $\mathrm{Ker}(\alpha) \supset \Gamma = \langle g_1, g_2 \rangle$. Actually, Γ is a subgroup of $\mathrm{Ker}(\alpha)$ of index two.

On the other hand, $\alpha(g_3)$ and $\alpha(g_4)$ act effectively on $PGL(2, \mathbb{C})$:

$$\begin{aligned} \alpha(g_3): \ (t_0, t_1) &\longmapsto (t_0', t_1') = (3t_0 + t_1, 18t_0 - 3t_1) \\ \alpha(g_4): \ (t_0, t_1) &\longmapsto (t_0', t_1') = (t_0, \zeta_3^2 t_1). \end{aligned}$$

Thus g_3 and g_4 induce the action on the Hesse pencil independent of its additive group structure. We can easily check the following relation

$$\alpha(g_3)^2 = \alpha(g_4)^3 = 1.$$

It follows that we have

$$\alpha(G_{216}) = \langle \alpha(g_3), \alpha(g_4) \rangle \simeq \mathcal{T},$$

where $\mathcal{T}$ is the tetrahedral group. Thus the group $\alpha(G_{216})$ acts on $PGL(2, \mathbb{C})$ as the permutations among the following 12 elements

$$\begin{aligned} &\lambda := \frac{t_1}{t_0}, \ \zeta_3\lambda, \ \zeta_3^2\lambda, \ \frac{18 - 3\lambda}{3 + \lambda}, \ \frac{18\zeta_3 - 3\zeta_3\lambda}{3 + \lambda}, \ \frac{18\zeta_3^2 - 3\zeta_3^2\lambda}{3 + \lambda}, \ \frac{18 - 3\zeta_3\lambda}{3 + \zeta_3\lambda}, \\ &\frac{18\zeta_3 - 3\zeta_3^2\lambda}{3 + \zeta_3\lambda}, \ \frac{18\zeta_3^2 - 3\lambda}{3 + \zeta_3\lambda}, \ \frac{18 - 3\zeta_3^2\lambda}{3 + \zeta_3^2\lambda}, \ \frac{18\zeta_3 - 3\lambda}{3 + \zeta_3^2\lambda}, \ \frac{18\zeta_3^2 - 3\zeta_3\lambda}{3 + \zeta_3^2\lambda}. \end{aligned}$$

The Hesse pencil contains four singular members with multiplicity three corresponding to the following (t_0, t_1) [1]

$$(t_0, t_1) = (0, 1), \ (1, -3), \ (1, -3\zeta_3^2), \ (1, -3\zeta_3).$$

Denote these points by s_i $(i = 1, 2, 3, 4)$ in order. These s_i's are permuted by $\alpha(G_{216})$ as follows

$$\alpha(g_3): \ s_1 \longleftrightarrow s_2, \quad s_3 \longleftrightarrow s_4 \tag{1}$$

$$\alpha(g_4): \ s_2 \longrightarrow s_3 \longrightarrow s_4 \longrightarrow s_2 \quad (s_1 \text{ is fixed.}) \tag{2}$$

Thus s_i's can be corresponded to the vertices of the tetrahedron on which $\mathcal{T} \simeq \alpha(G_{216})$ acts.

Moreover, let

$$g_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then we have

$$g_3^2 = (g_4 g_0)^3 = g_0.$$

Therefore, we obtain

$$G_{216}/\Gamma = \langle g_3, g_4 \rangle \simeq \tilde{\mathcal{T}},$$

where $\tilde{\mathcal{T}}$ is the binary tetrahedral group. Since $\tilde{\mathcal{T}}$ is isomorphic to $SL(2,\mathbb{F}_3)$, we obtain the semi-direct product decomposition $G_{216} \simeq \Gamma \rtimes SL(2,\mathbb{F}_3)$.

The level-three theta functions $\theta_0(z,\tau)$, $\theta_1(z,\tau)$, and $\theta_2(z,\tau)$ are defined by using the theta function $\vartheta_{(a,b)}(z,\tau)$ with characteristics:

$$\theta_k(z,\tau) := \vartheta_{\left(\frac{k}{3}-\frac{1}{6},\frac{3}{2}\right)}(3z,3\tau) = \sum_{n\in\mathbb{Z}} e^{3\pi i\left(n+\frac{k}{3}-\frac{1}{6}\right)^2\tau} e^{6\pi i\left(n+\frac{k}{3}-\frac{1}{6}\right)\left(z+\frac{1}{2}\right)} \quad (k=0,1,2),$$

where $z \in \mathbb{C}$ and $\tau \in \mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}\tau > 0\}$. Fixing $\tau \in \mathbb{H}$, we abbreviate $\theta_k(z,\tau)$ and $\theta_k(0,\tau)$ as $\theta_k(z)$ and θ_k for $k=0,1,2$, respectively. We can easily see that the following holds

$$\theta_0 = -\theta_1 \qquad \theta_2 = 0. \tag{3}$$

Let $L_\tau := (-\tau)\mathbb{Z} + (3\tau+1)\mathbb{Z}$ be a lattice in $\mathbb{C}$. Consider a map $\varphi : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$,

$$\varphi :\ z \longmapsto (\theta_2(z), \theta_0(z), \theta_1(z)).$$

This induces an isomorphism from the complex torus $\mathbb{C}/L_\tau$ to the Hesse cubic curve $E_{\theta_2',6\theta_0'}$. It also induces the additive group structure on $E_{\theta_2',6\theta_0'}$ from $\mathbb{C}/L_\tau$ through the addition formulae for the level-three theta functions [2]; let (x_0,x_1,x_2) and (x_0',x_1',x_2') be points on $E_{\theta_2',6\theta_0'}$, then the addition $(x_0,x_1,x_2)+(x_0',x_1',x_2')$ of the points is given as follows

$$(x_0,x_1,x_2)+(x_0',x_1',x_2') = (x_1x_2{x_2'}^2 - x_0^2x_0'x_1',\, x_0x_1{x_1'}^2 - x_2^2x_0'x_2',\, x_0x_2{x_0'}^2 - x_1^2x_1'x_2'). \tag{4}$$

The relation (3) implies that the unit of addition on $E_{\theta_2',6\theta_0'}$ induced by φ is p_0:

$$\varphi :\ 0 \longmapsto (\theta_2,\theta_0,\theta_1) = (0,1,-1) = p_0.$$

By using (4), we see that the actions of g_1 and g_2 on $E_{\theta_2',6\theta_0'}$ can be realized as the additions with p_6 and p_1, respectively

$$\begin{aligned} (x_0,x_1,x_2) &\overset{g_1}{\longmapsto} (x_1,x_2,x_0) = (x_0,x_1,x_2)+p_6 \\ (x_0,x_1,x_2) &\overset{g_2}{\longmapsto} (x_0,\zeta_3x_1,\zeta_3^2x_2) = (x_0,x_1,x_2)+p_1. \end{aligned}$$

Take the following representatives z_{0k}, z_{k1}, z_{k2} of the zeros of $\theta_k(z)$ in $\mathbb{C}/L_\tau$ for $k=0,1,2$

$$\begin{pmatrix} z_{20} & z_{21} & z_{22} \\ z_{00} & z_{01} & z_{02} \\ z_{10} & z_{11} & z_{12} \end{pmatrix} = \begin{pmatrix} 0 & \tau+\frac{1}{3} & 2\tau+\frac{2}{3} \\ -\frac{\tau}{3} & \frac{2\tau}{3}+\frac{1}{3} & \frac{5\tau}{3}+\frac{2}{3} \\ -\frac{2\tau}{3} & \frac{\tau}{3}+\frac{1}{3} & \frac{4\tau}{3}+\frac{2}{3} \end{pmatrix}.$$

Then these nine zeros are mapped into the nine inflection points on $E_{\theta_2',6\theta_0'}$ by φ, respectively:

$$\varphi :\ \begin{matrix} z_{20} & z_{21} & z_{22} \\ z_{00} & z_{01} & z_{02} \\ z_{10} & z_{11} & z_{12} \end{matrix} \quad \longmapsto \quad \begin{matrix} p_0 & p_1 & p_2 \\ p_3 & p_4 & p_5 \\ p_6 & p_7 & p_8 \end{matrix}.$$

Let us tropicalize the Hesse pencil. For the defining polynomial $f(x_0,x_1,x_2;t_0,t_1)$ of the Hesse cubic curve, we apply the procedure of tropicalization. Replacing $+$ and $\times$ with max and $+$ respectively, the polynomial $f(x_0,x_1,x_2;t_0,t_1)$ reduces to

$$\tilde{f}(\tilde{x}_0,\tilde{x}_1,\tilde{x}_2;\tilde{t}_0,\tilde{t}_1) = \max\left(\tilde{t}_0+3\tilde{x}_0, \tilde{t}_0+3\tilde{x}_1, \tilde{t}_0+3\tilde{x}_2, \tilde{t}_1+\tilde{x}_0+\tilde{x}_1+\tilde{x}_2\right).$$

In order to distinguish tropical variables form original ones, we ornament them with ˜.

Let (t_0, t_1) be a point in $\mathbb{P}^{1,trop}$, the tropical projective line. Then $\tilde{f}$ can be regarded as a function $\tilde{f} : \mathbb{P}^{2,trop} \to \mathbb{T}$, where $\mathbb{P}^{2,trop}$ is the tropical projective plane and $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$ is the tropical semi-field. The tropical Hesse curve is the set of points such that the function $\tilde{f}$ is not differentiable. We denote the tropical Hesse curve by $C_{\tilde{t}_0, \tilde{t}_1}$. Upon introduction of the inhomogeneous coordinate $(X := \tilde{x}_1 - \tilde{x}_0, Y := \tilde{x}_2 - \tilde{x}_0) \in \mathbb{P}^{2,trop}$ and $K := \tilde{t}_1 - \tilde{t}_0 \in \mathbb{P}^{1,trop}$ the tropical Hesse curve is denoted by C_K and is given by the tropical polynomial

$$F(X, Y; K) := \max(3X, 3Y, 0, K + X + Y).$$

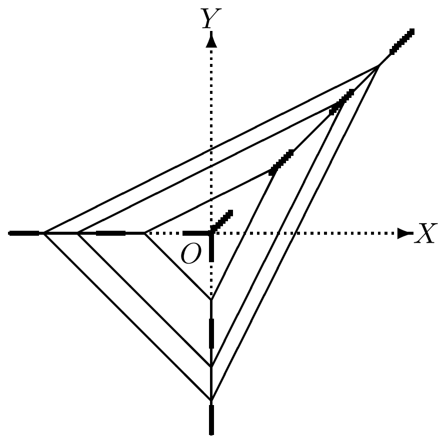

Figure 2: The curves drawn with solid lines are the regular members and the one with broken line is the singular member of the tropical Hesse pencil.

Figure 2 shows the tropical Hesse curves. The one-dimensional linear system $\{C_K\}_{K \in \mathbb{P}^{1,trop}}$ consisting of the tropical Hesse curves is called the tropical Hesse pencil. The complement of the tentacles, i.e., the finite part, of C_K is denoted by $\bar{C}_K$. We denote the vertices whose coordinates are (K, K), $(-K, 0)$, and $(0, -K)$ by V_1, V_2, and V_3, respectively.

Let K and ε be positive numbers. Let us fix τ:

$$\tau = -\frac{3K}{9K + 2\pi i \varepsilon}.$$

Then the complex torus $\mathbb{C}/L_\tau$ converges into $J(C_K)$ in the limit $\varepsilon \to 0$ with respect to the Hausdorff metric [3]. Let us introduce a map $\tilde{\varphi} : J(C_K) \to \mathbb{R}^2 \subset \mathbb{P}^{2,trop}$,

$$\tilde{\varphi} : \quad u \longmapsto (\tilde{c}(u), \tilde{s}(u)),$$

where we define

$$\tilde{c}(u) := -\frac{9K}{2}\left\{\left(\left(\frac{u-K}{3K} - \frac{1}{2}\right)\right)\right\}^2 + \frac{9K}{2}\left\{\left(\left(\frac{u-3K}{3K} - \frac{1}{2}\right)\right)\right\}^2$$
$$\tilde{s}(u) := -\frac{9K}{2}\left\{\left(\left(\frac{u-2K}{3K} - \frac{1}{2}\right)\right)\right\}^2 + \frac{9K}{2}\left\{\left(\left(\frac{u-3K}{3K} - \frac{1}{2}\right)\right)\right\}^2,$$

and $((u)) := u - \mathrm{Floor}(u)$. This map induces an isomorphism $\bar{C}_K \simeq J(C_K)$ [2, 3], where $J(C_K)$ is the tropical Jacobian of C_K:

$$J(C_K) := \mathbb{R}/3K\mathbb{Z} = \{u \in \mathbb{R} \mid 0 \le u < 3K\}.$$

Thus $\tilde{\varphi}$ induces additive group structure on $\bar{C}_K$ equipped with the unit of addition $V_1 = \tilde{\varphi}(0)$ form $J(C_K)$. The addition formula for $\bar{C}_K$ is explicitly given in [3].

The piecewise linear functions $\tilde{c}(u)$ and $\tilde{s}(u)$ are periodic with period $3K$ and are the ultradiscretization of the elliptic functions $c(z) := \theta_0(z, \tau)/\theta_2(z, \tau)$ and $s(z) := \theta_1(z, \tau)/\theta_2(z, \tau)$, respectively [2, 3]. In the procedure of ultradiscretization, we assume $u \in \mathbb{R}$ and

$$z = \frac{(1 - i\xi_\varepsilon)\, u}{9K},$$

where $\xi_\varepsilon = 2\pi\varepsilon/9K$, and take the limit $\varepsilon \to 0$. In terms of the variable u, we put the limit of zeros z_{kj} $(k, j = 0, 1, 2)$ of the level-three theta functions as follows

$$u_2 := \lim_{\varepsilon \to 0} 9K z_{20} = \lim_{\varepsilon \to 0} 9K z_{21} = \lim_{\varepsilon \to 0} 9K z_{22} = 0$$
$$u_0 := \lim_{\varepsilon \to 0} 9K z_{00} = \lim_{\varepsilon \to 0} 9K z_{01} = \lim_{\varepsilon \to 0} 9K z_{02} = K$$
$$u_1 := \lim_{\varepsilon \to 0} 9K z_{10} = \lim_{\varepsilon \to 0} 9K z_{11} = \lim_{\varepsilon \to 0} 9K z_{12} = 2K,$$

where it should be noted that $\tau \to -1/3$ in the limit $\varepsilon \to 0$.

Consider a map $\eta : E_{\theta_2', 6\theta_0'} \to \bar{C}_K$ so defined that the diagram commute

$$\begin{array}{ccc} \mathbb{C}/L_\tau & \xrightarrow{\varepsilon \to 0} & J(C_K) \\ \varphi \big\downarrow & & \big\downarrow \tilde{\varphi} \\ E_{\theta_2', 6\theta_0'} & \xrightarrow{\eta} & \bar{C}_K. \end{array}$$

The inflection points of $E_{\theta_2', 6\theta_0'}$ are mapped into the vertices of $\bar{C}_K$ by η as follows

$$\eta : \ p_0, \ p_1, \ p_2 \xmapsto{\varphi^{-1}} z_{20}, \ z_{21}, \ z_{22} \xrightarrow{\varepsilon \to 0} u_2 \xmapsto{\tilde{\varphi}} V_1 \tag{5}$$

$$\eta : \ p_3, \ p_4, \ p_5 \xmapsto{\varphi^{-1}} z_{00}, \ z_{01}, \ z_{02} \xrightarrow{\varepsilon \to 0} u_0 \xmapsto{\tilde{\varphi}} V_2 \tag{6}$$

$$\eta : \ p_6, \ p_7, \ p_8 \xmapsto{\varphi^{-1}} z_{10}, \ z_{11}, \ z_{12} \xrightarrow{\varepsilon \to 0} u_1 \xmapsto{\tilde{\varphi}} V_3. \tag{7}$$

Now we investigate the tropical counterpart of the Hessian group $G_{216} \simeq \Gamma \rtimes SL(2, \mathbb{F}_3)$. At first we consider $\Gamma \simeq (\mathbb{Z}/3\mathbb{Z})^2$. Note that $\Gamma = \langle g_1, g_2 \rangle$ and the actions of g_1 and g_2 on E_{t_0,t_1} is realized as the additions with p_6 and p_1, respectively. Moreover, the group generated by the additions with p_6 and p_1 is nothing but $E_{t_0,t_1}[3]$, the group of three torsion points on E_{t_0,t_1}.

The correspondence (5), (6), and (7) in terms of η tells us that the addition with p_6 corresponds to that with V_3 on C_K, while that with p_1 vanishes in the limit $\varepsilon \to 0$. (Note that V_1 is the unit of addition on C_K.) Since the addition with p_3 (resp. V_2) is equivalent to that with $2p_6$ (resp. $2V_3$), the tropical analogue of Γ consists of the addition with V_3. Actually, it is the group $C_K[3] = \langle V_3 \rangle$ of three torsion points on C_K, which is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. We denote the tropical analogue of a group G by $trop(G)$:

$$trop(\Gamma) \simeq \mathbb{Z}/3\mathbb{Z}.$$

The addition with V_3 is explicitly computed as follows

$$(X, Y) \uplus V_3 = (X, Y) \uplus (0, -K) = (Y - X, -X),$$

where we denote the addition on C_K by $\uplus$ and apply the addition formula [3]

$$\begin{aligned} (X, Y) \uplus (X', Y') = \big(&\max\left(Y, 2X + X' + Y'\right) - \max\left(X + 2X', 2Y + Y'\right), \\ &\max\left(X + Y + 2Y', X'\right) - \max\left(X + 2X', 2Y + Y'\right)\big). \end{aligned}$$

The group $trop(\Gamma)$ can also be obtained by applying the procedure of ultradiscretization directly to g_1 and g_2. Let us consider the inhomogeneous coordinate $(x := x_1/x_0, y := x_2/x_0)$ of $\mathbb{P}^2(\mathbb{C})$. Let $g_1 : (x, y) \mapsto (x', y')$ and $g_2 : (x, y) \mapsto (x'', y'')$. Then we have

$$\left(|x'|, |y'|\right) = \left(\frac{|y|}{|x|}, \frac{1}{|x|}\right) \qquad \left(|x''|, |y''|\right) = (|x|, |y|).$$

Replacing $|x|$ and $|y|$ with $e^{X/\varepsilon}$ and $e^{Y/\varepsilon}$ respectively and taking the limit $\varepsilon \to 0$, we obtain

$$(X, Y) \xmapsto{\tilde{g}_1} (Y - X, -X) = (X, Y) \uplus V_3 \qquad (X, Y) \xmapsto{\tilde{g}_2} (X, Y),$$

where we denote the action on $\mathbb{P}^{2,trop}$ induced form g_1 and g_2 by $\tilde{g}_1$ and $\tilde{g}_2$, respectively.

Next we consider $\alpha(G_{216}) \simeq \mathcal{T}$. Remember that each singular member E_{s_i} $(i = 1, 2, 3, 4)$ in the Hesse pencil corresponds to the vertex of the tetrahedron on which $\mathcal{T}$ acts (see (1) and (2)). The singular members of the tropical Hesse pencil are C_∞ and C_0 which are the tropicalization of E_{s_1}

and E_{s_i}, $(i = 2, 3, 4)$, respectively [3]. Thus the action of $\alpha(g_3)$, which permutes s_1 and s_3 with s_2 and s_4 respectively, must vanish; while the action of $\alpha(g_4)$, which fixes s_1 and permutes s_2, s_3, and s_4 cyclically, reduces to the action fixing both C_0 and C_∞. Therefore, we have

$$\mathit{trop}\left(\alpha(G_{216})\right) \simeq \mathit{trop}\left(\mathcal{T}\right) \simeq \langle 1 \rangle .$$

Thus the tropical analogue of the Hessian group fixes each member of the tropical Hesse pencil.

Furthermore, we consider the tropicalization of $g_0 = g_3^2$. We ultradiscretize g_0 directly as well as g_1 and g_2. The action of g_0 on $\mathbb{P}^2(\mathbb{C})$ is given as

$$(x, y) \overset{g_0}{\longmapsto} (y, x)$$

in the inhomogeneous coordinate. It follows that we have

$$(X, Y) \overset{\tilde{g}_0}{\longmapsto} (Y, X) ,$$

where (X, Y) is the inhomogeneous coordinate of $\mathbb{P}^{2,trop}$ and $\tilde{g}_0$ is the action on $\mathbb{P}^{2,trop}$ induced from that of g_0 by applying the procedure of ultradiscretization. Thus we conclude that the tropical analogue of $\tilde{\mathcal{T}} \simeq \langle g_3, g_4 \rangle = G_{216}/\Gamma$ is the group of order two generated by $\tilde{g}_0$:

$$\mathit{trop}\left(\tilde{\mathcal{T}}\right) \simeq \langle \tilde{g}_0 \rangle \simeq \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\rangle \subset SL(2, \mathbb{F}_3).$$

We then obtain the following theorem concerning the tropical analogue of G_{216}.

Theorem 1 The dihedral group $\mathcal{D}_3$ of degree three,

$$\mathcal{D}_3 = \langle \tilde{g}_0, \tilde{g}_1 \rangle \simeq (\mathbb{Z}/3\mathbb{Z}) \rtimes \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\rangle$$

where $\tilde{g}_0, \tilde{g}_1 \in PGL(3, \mathbb{T})$ satisfy $\tilde{g}_0^2 = \tilde{g}_1^3 = (\tilde{g}_0\tilde{g}_1)^2 = 1$, is the group of linear automorphisms acting on the tropical Hesse pencil[1]. The action of $\tilde{g}_0$ on each curve of the pencil is realized as the reflection with respect to the line $Y = X$ passing through the vertex V_1; and the action of $\tilde{g}_1$ on each curve is realized as the addition with V_3.

In this paper, we consider linear automorphisms acting on the Hesse pencil only. To investigate a tropical analogue of the Cremona group, the group of birational automorphisms acting on the Hesse pencil, is a further problem.

References

[1] Artebani M and Dolgachev I, *Preprint* arXiv:math/0611590v3 (2006)

[2] Kajiwara K, Kaneko M, Nobe A and Tsuda T, *Kyushu J. Math.* **63** (2009) 315-338

[3] Nobe A, *Preprint*, to appear in *RIMS Kokyuroku*

[4] Shaub H C and Schoonmaker H E, *Am. Math. Mon.* **38** (1931) 388-393

[1] Identifying $\gamma_i \simeq (\gamma_i, 0) \in (\mathbb{Z}/3\mathbb{Z})^2$, we define the multiplication of $(\gamma_1, m_1), (\gamma_2, m_2) \in (\mathbb{Z}/3\mathbb{Z}) \rtimes \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\rangle$ by

$$(\gamma_1, m_1) \cdot (\gamma_2, m_2) = (\gamma_1 + m_1\gamma_2, m_1 m_2).$$